(34) Q.No → If $\sin u = \dfrac{x^3+y^3}{x-y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u$.

Ans → $\because \sin u = \dfrac{x^3+y^3}{x+y} = \dfrac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)} = x^2 f\left(\frac{y}{x}\right) = v$ — (1)

Hence v is a homogeneous function of x and y of degree two

Hence, from Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 2\times v$$

Now, $v = \sin u$ — (2)

$\therefore$ Diff. (2) partially w.r.t. x we have,

$$\frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x}$$

$$x\frac{\partial v}{\partial x} = x\cos u\frac{\partial u}{\partial x} \quad \text{— (a)}$$

Similarly $$y\frac{\partial v}{\partial y} = y\cos u\frac{\partial u}{\partial y} \quad \text{— (b)}$$

(a) + (b)

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$2v = 2\sin u = \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\frac{2\sin u}{\cos u} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$2\tan u = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$ proved.

QN.→ ~~If~~ $u = \tan^{-1}\frac{y}{x}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Ans.→ $u = \tan^{-1}\frac{y}{x}$

Diff. partially w.r.t. y keeping x constant

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \times 1$$

$$\frac{\partial u}{\partial y} = \frac{x^2}{x^2+y^2} \times \frac{1}{\cancel{x}} = \frac{x}{x^2+y^2}$$

Again diff. partially w.r.t. y, we have,

$$\frac{\partial^2 u}{\partial y^2} = x\left[\frac{\partial}{\partial x}(x^2+y^2)^{-1}\right]$$

$$= x \times -1(x^2+y^2)^{-2} \times 2y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \quad \text{——— (1)}$$

$$u = \tan^{-1}\frac{y}{x}$$

Diff. partially w.r.t. x keeping y constant

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial x} = -\frac{\cancel{x^2}}{x^2+y^2} \times \frac{y}{\cancel{x^2}} = -\frac{y}{x^2+y^2}$$

Again diff. partially w.r.t. x, we have

$$\frac{\partial^2 u}{\partial x^2} = y\left[\frac{\partial}{\partial x}(x^2+y^2)^{-1}\right]$$

$$= -y \times -1(x^2+y^2)^{-2} \times 2x$$

$$= \frac{2xy}{(x^2+y^2)^2} \quad \text{--- (2)}$$

(2) + (1)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{\frac{2xy}{(x^2+y^2)^2}} - \cancel{\frac{2xy}{(x^2+y^2)^2}} = 0$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ Proved.}$$

(37) QN. → verify Euler's theorem for the ~~exa~~ expression $x^n \sin\frac{y}{x}$.

Ans. → Here, u is a homogeneous function of x and y of degree n.

Hence, to verify Euler's theorem

I have to prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$$u = x^n \sin\frac{y}{x} \quad \text{--- (A)}$$

Diff. partially w.r.t. x, keeping y constant

$$\frac{\partial u}{\partial x} = x^n \cos\frac{y}{x} \times y \times -\frac{1}{x^2} + \sin\frac{y}{x} \times n x^{n-1}$$

$$\therefore x\frac{\partial u}{\partial x} = -x^{n+1}\cos\frac{y}{x} \times \frac{y}{x^2} + n\sin\frac{y}{x} \cdot x^n$$

$$x\frac{\partial u}{\partial x} = -y x^{n-1}\cos\frac{y}{x} + n x^n \cdot \sin\frac{y}{x} \quad \text{—(1)}$$

Again diff. (A) partially w.r.t. y keeping x const

$$\frac{\partial u}{\partial y} = x^n \times \cos\frac{y}{x} \times \frac{1}{x} \times 1$$

$$\frac{\partial u}{\partial y} = x^{n-1}\cos\frac{y}{x}$$

$$\therefore y\frac{\partial u}{\partial y} = y x^{n-1}\cos\frac{y}{x} \quad \text{—(2)}$$

(1) + (2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{-y x^{n-1}\cos\frac{y}{x}} + n x^n \sin\frac{y}{x} + \cancel{y x^{n-1}\cos\frac{y}{x}}$$

$$= n x^n \sin\frac{y}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Hence, Euler's theorm is verification